JOURNAL OF COMPUTATIONAL PHYSICS 99, 337–340 (1992)

NOTE

Two-Point Quasi-Fractional Approximations to the Airy Function Ai(x)

I. INTRODUCTION

The Airy function appears very often in several areas of physics such as quantum mechanics, electrodynamics, plasma physics, etc. [1–3]. Although the WKB method provides a good approximation to the Airy function for large values of the independent variable, a simple expression valid throughout for every positive (or alternatively, negative) real x is not available. The two-point quasi-fractional approximation procedure recently published [4–7] shows a way to obtain simple approximations valid throughout the region of validity.

The main idea is to establish a form of approximation such that the singularities of the approximation are coincident with those of the exact function in all the range of interest. In previous papers [4–6], the quasi-fractional approximations are defined in terms of the usual independent variable. However, in the Airy case, to obtain the coincidence of the singularities a change of variable has to be introduced and then the correct quasi-fractional form is found. The form of the approximation is different for the negative region than for the positive one. However, in a small region around the origin both forms are valid.

In this paper we have obtained the simplest approximation using first-degree polynomials, the accuracy is such that the approximated and the exact curves are indistinguishable when plotted on graph paper of standard size. More than two decimal place accuracy is obtained for all values of x. Higher order precision routines for Airy and their derivatives exist [8, 9].

However, one important advantage of the approximations obtained here is that they can be derived or used inside integrands by substituting the exact function in the whole real axis and the accuracy will suffice for many applications. Furthermore, the numerical computation of the approximation can be obtained quickly and economically, even using a pocket calculator. While it is true that the power series for the exact Ai(x) is convergent for all values of x and can be used for the numerical computation to any degree of accuracy; the number of terms to be used are numerous for intermediate or large values of x. Also the accuracy of the asymptotic expansions depend strongly on the value of x and it is not easy to find upper bounds. Our approximations are obtained using the leading terms of the asymptotic expansion and a suitable number of terms of the power series. However, the accuracy of the quasi-fractional approximation is higher than the accuracy of either the power or the asymptotic series calculated with the same number of terms as our approximations.

In Section II we discuss the procedure to be used and the determination of the right independent variable. In Section III we determine the values of the parameters of the approximation and we discuss the results and the accuracy obtained as well as that of the WKB method (leading term of the asymptotic expansion) and partial power series. Section IV is devoted to general discussion and conclusions.

II. THEORETICAL TREATMENT

The Airy function Ai(x) has an essential singularity at infinity. The asymptotic expansion shows the Stokes phenomenon, and the form is different for positive x than for negative x [7, 10–12]. Besides, the power series is a sum of two series in terms of the variable x^3. For real positive x the leading term is

$$(1/2\pi^{1/2})\, x^{-1/4} \exp(-2x^{3/2}/3). \tag{1}$$

From this expression and from the conditions imposed by the power expansion it could be concluded that a suitable variable is $x^{3/2}$. However, this variable presents the problem of a branch point at zero, which is not present in the Airy exact function. Since this point is in the region of interest we have to slightly modify this variable to avoid this inconvenience. A simple way to do this is to choose the variable $\sqrt{\lambda + x^3}$, where λ can be any positive number. This variable is inadequate, however, for negative x. Therefore a better choice valid for both regions is

$$z = \sqrt{\lambda_i + |x|^3}, \tag{2}$$

where λ_1 and λ_2 correspond to positive and negative values of x, respectively.

Considering now the restrictions imposed by the power

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series and by the asymptotic expansion, we have found the general form

$$y_n = \left(\sum_{i=0}^{n} p_i z^i + (x/z^{2/3}) \sum_{i=0}^{n} P_i z^i \right) \frac{\exp(-2z/3)}{z^{(n+1/6)}} \tag{3}$$

which has the right series expansion and also an adequate asymptotic behavior. In previous papers [4–6] we define the denominator in terms of q-parameters to be determined together with the p-parameters of the numerator. However, this procedure sometimes causes undesirable zeros in the denominator without substantially improving the accuracy. The procedure presented here is more general than the previous one; we fix the denominator in the simplest way.

The form of the approximation is valid also for small negative values of x which are not too close to the point $x = -(\lambda_1)^{1/3}$ (the branch point). For large negative values of x, we have to consider a different form due to the Stokes phenomenon. An analysis similar to the one mentioned above for positive x, leads to an approximation of the form

$$y_n = \sum_{i=0}^{n} q_i z^i \operatorname{Cos}(2\,|x|^{3/2}/3)/z^{n+1/6} + x \sum_{i=0}^{n} Q_i z^i \operatorname{Sin}(2\,|x|^{3/2}/3)/(|x|^{3/2}\, z^{n-1/6}). \tag{4}$$

Here we can choose directly $\operatorname{Sin}(2\,|x|^{3/2}/3)$ and $\operatorname{Cos}(2\,|x|^{3/2}/3)$ instead of $\operatorname{Sin}(z)$ and $\operatorname{Cos}(z)$, because the functions $|x|^{-3/2}\operatorname{Sin}(2\,|x|^{3/2}/3)$ and $\operatorname{Cos}(2\,|x|^{3/2}/3)$ are even functions. Thus we have only even powers and therefore the branch point disappears.

III. RESULTS

The simplest case $n = 1$, leads to the expressions

$$Y_1(x) = \left(\frac{p_0 + p_1 z}{z^{7/6}} + \frac{P_0 + P_1 z}{z^{11/6}} \right) \times \exp(-2z/3) \quad \text{if} \quad x \geqslant 0 \tag{5}$$

and

$$Y_1(x) = \left(\frac{q_0 + q_1 z}{z^{7/6}} \operatorname{Cos}(2\,|x|^{3/2}/3) + x\, \frac{Q_0 + Q_1 z}{z^{5/6}} \frac{\operatorname{Sin}(2\,|x|^{3/2}/3)}{|x|^{3/2}} \right) \tag{6}$$

if $x \leqslant 0$.

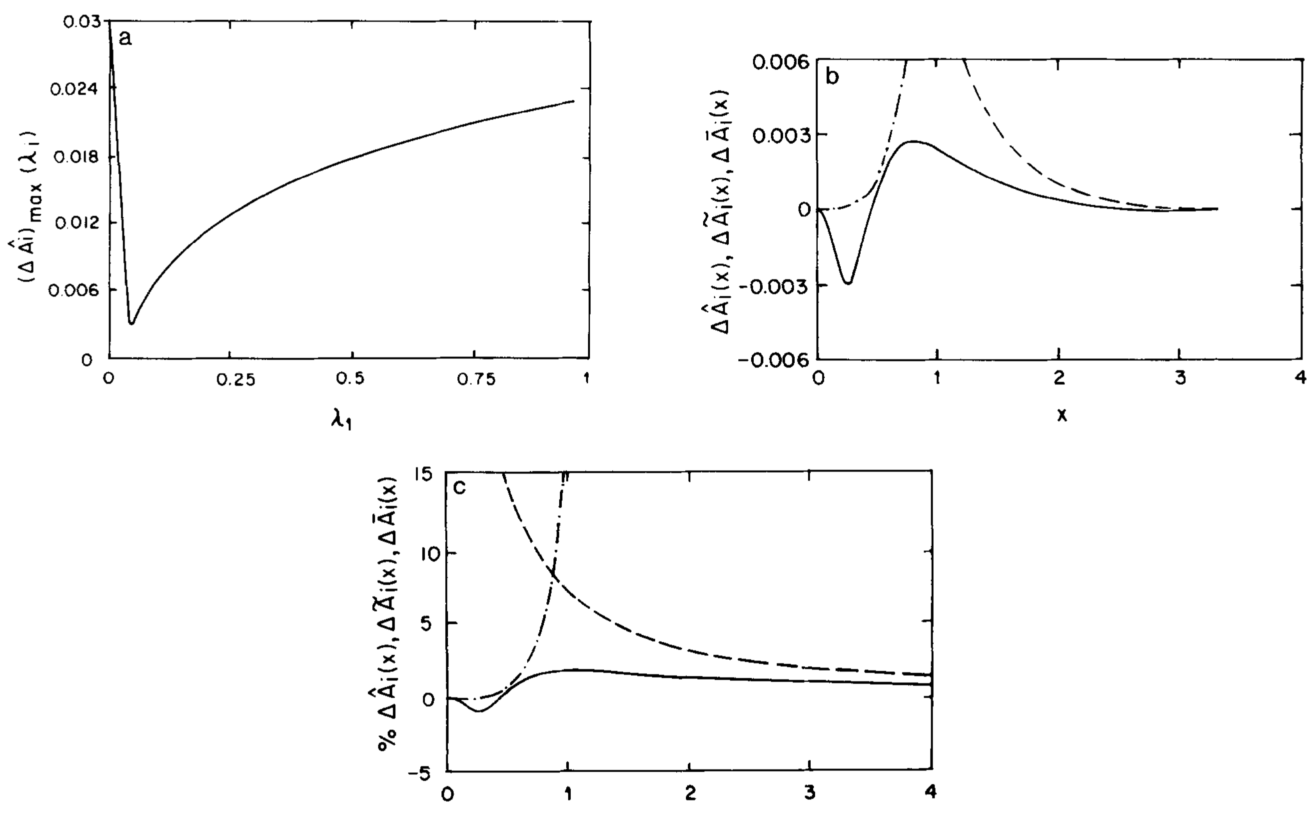

FIG. 1. (a) The maximum error as a function of λ_1 for $x \geqslant 0$. (b) The absolute errors for our approximation with $n = 1$ (full line), the WKB approximation (dashed line), and the partial power series (point-dash line) for $x \geqslant 0$. (c) The relative errors for our approximation with $n = 1$ (full line), for the WKB approximation (dashed line), and for the partial power series (point-dash line) for $x \geqslant 0$.

The asymptotic condition leads to the equations

$$p_1 + P_1 = 1/2\pi^{1/2}; \quad q_1 = 1/(2\pi)^{1/2}; \quad Q_1 = -q_1. \tag{7}$$

The first fraction in Eq. (5) will give the zeroth (a_0) and the third ($a_0/6$) degree terms of the Airy power series. The last equation for determining the four parameters p_0, p_1, P_0, and P_1 is obtained by equating the first-degree terms of Eq. (5) and of the Airy power series (a_1). In this way all the p and P parameters are determined as functions of the parameter λ_1:

$$\begin{aligned} p_0 + \sqrt{\lambda_1}\, p_1 &= \lambda_1^{7/12} \exp(\tfrac{2}{3}\sqrt{\lambda_1})\, \mathrm{Ai}(0) \\ P_0 + \sqrt{\lambda_1}\, P_1 &= \lambda_1^{11/12} \exp(\tfrac{2}{3}\sqrt{\lambda_1})\, \mathrm{Ai}'(0) \\ (7 + 4\sqrt{\lambda_1})\, p_0 &+ (1 + 4\sqrt{\lambda_1})\, \lambda_1 p_1 \\ &= -2\lambda_1^{19/12} \exp(\tfrac{2}{3}\sqrt{\lambda_1})\, \mathrm{Ai}(0). \end{aligned} \tag{8}$$

For the negative values, we have two equations coming from the asymptotic condition. Therefore we have to use only the zeroth (a_0) and first (a_1) degree terms of the Airy power series. Thus we obtain

$$\begin{aligned} q_0 + \sqrt{\lambda_2}\, q_1 &= \lambda_2^{7/12}\, \mathrm{Ai}(0) \\ Q_0 + \sqrt{\lambda_2}\, Q_1 &= (3/2)\, \lambda_2^{5/12}\, \mathrm{Ai}'(0). \end{aligned} \tag{9}$$

If we choose a given value of λ_1, i.e., $\lambda_1 = 1$, we can determine the error as a function of x. The approximations are very precise for small and large values of x. The maximum error occurs at intermediate values of x, i.e., in the region between 0.5 and 5. In order to determine the best λ_i we decide to plot this maximum error as a function of λ_i and to select the λ_i which gives the least maximum error. The results are shown in Fig. 1a for the positive region. In the negative region the pattern is similar. The best values obtained are $\lambda_1 \cong 0.0425$ and $\lambda_2 \cong 0.37$. Using these values for λ_1 and λ_2, we obtain

$$\begin{aligned} p_0 &= -0.002800908; & p_1 &= 0.326662423; \\ q_0 &= -0.043883564; & q_1 &= 0.3989422; \\ P_0 &= -0.007232251; & P_1 &= -0.044567423 \\ Q_0 &= -0.013883003; & Q_1 &= -0.3989422. \end{aligned}$$

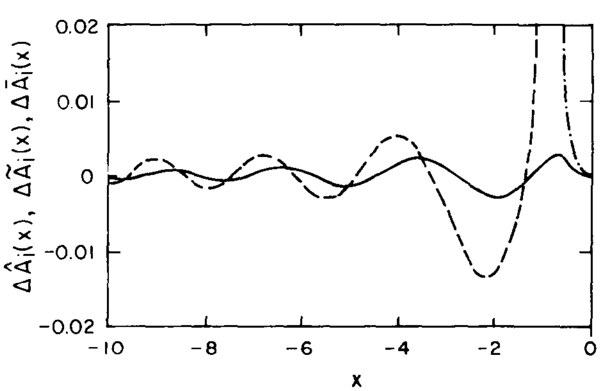

FIG. 2. The absolute errors for our approximation with $n = 1$ (full line), the WKB approximation (dashed line), and the partial power series (point-dash line) for $x \leqslant 0$.

From these values we can compute the approximations. Both curves are coincident in the scale drawing. In Fig. 1b and 2 we compare the absolute errors of our approximations (full line) with those obtained from the WKB method (dashed line) and from the partial power series (point-dash line). In Fig. 1c we show the relative error of the preceding functions (for positive x). Since in the negative side the Airy function has zeros, the relative errors would become infinite at them. Therefore plotting relative errors is not suitable. The WKB method (leading term of the asymptotic expansion) clearly fails in the region $|x| < 1.2$. Similarly, the partial power series fails for values such that $|x| > 0.8$. The accuracy of the function is fairly high for small and large values of x. The largest errors occur around $x \cong \pm 1$; their values are 0.003 (1 %) at $x = -1.9$ and $x = 0.25$.

IV. DISCUSSION

The two-point quasifractional method can be extended to some functions—as Airy's—by using suitable independent variables (e.g., $z = \sqrt{\lambda_i + |x|^3}$) in the fractional part of the quasifractional approximation. The forms of the approximation are, in the Airy case, different for the negative axis than for the positive one, due to the Stokes phenomenon. These forms have been determined by considering asymptotic expansions. In this paper the parameters of the denominator are fixed beforehand, avoiding undesirable zeros in the denominator. The simplest approximation is obtained with a first-degree polynomial combined with exponential and trigonometric functions. The accuracy is high for such simple approximations and no difference with the exact function can be noted in standard size plotting. More than two decimal place accuracy is obtained for all the real range of the variable. The accuracy for very small and very large values of x is much better (three, four, and even more decimal places). The largest error is 0.003 (1 %) at $x = -1.9$ and $x = 0.25$.

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Received July 20, 1989; revised January 29, 1991

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